Question No. 6061-1P

Show that the functions $\psi_1(x) = 1$ and $\psi_2(x) = x$ are orthogonal on the interval $-1 < x < 1$, and determine constants A and B such that the function

$$\psi_3(x) = 1 + Ax + Bx^2$$

is orthogonal to both ψ_1 and ψ_2 on that interval.

Solution:

It is given that, $\Psi_1(\mathrm{x}) = 1$ and $\Psi_2(\mathrm{x}) = x$ within the interval $-1 < x < 1$

To show they are orthogonal the integral of the product of these fuctions within (-1, 1) must be 0.

Integral of the product of these fuctions:

$$\begin{aligned}
\int_{-1}^{1} \left(\Psi_1(\mathrm{x}) \times \Psi_1(\mathrm{x})\right) dx &= \int_{-1}^{1} (1 \times x) dx \\
&= \int_{-1}^{1} x dx \\
&= \left[\frac{x^2}{2}\right]_{-1}^{1} \\
&= \frac{1^2}{2} - \frac{(-1)^2}{2} \\
&= \frac{1}{2} - \frac{1}{2} \\
&= 0
\end{aligned}$$

Since $\int_{-1}^{1} \left(\Psi_1(\mathrm{x}) \times \Psi_1(\mathrm{x})\right) dx = 0$, therefore $\Psi_1(\mathrm{x}) = 1$ and $\Psi_2(\mathrm{x}) = x$ are orthogonal within the interval.

Now it is given that, $\Psi_3(\mathrm{x}) = 1 + Ax + Bx^2$ is orthogonal to $\Psi_1(\mathrm{x}) = 1$ and $\Psi_2(\mathrm{x}) = x$.

When $\Psi_3(\mathrm{x}) = 1 + Ax + Bx^2$ is orthogonal to $\Psi_1(\mathrm{x}) = 1$, the integral of the product of these fuctions within (-1, 1) will be 0. Therefore,

$$\int_{-1}^{1} \left(\Psi_1(x) \times \Psi_3(x)\right) dx = 0$$

$$\int_{-1}^{1} \left(1 \times (1 + Ax + Bx^2)\right) dx = 0$$

$$\int_{-1}^{1} (1 + Ax + Bx^2) dx = 0$$

$$\left[x + A\frac{x^2}{2} + B\frac{x^3}{3}\right]_{-1}^{1} = 0$$

$$\left(1 - (-1)\right) + \frac{A}{2}\left(1^2 - (-1)^2\right) + \frac{B}{3}\left(1^3 - (-1)^3\right) = 0$$

$$(1 + 1) + \frac{A}{2}(1 - 1) + \frac{B}{3}(1 + 1) = 0$$

$$2 + \frac{A}{2} \times 0 + \frac{B}{3} \times 2 = 0$$

$$2 + \frac{2B}{3} = 0$$

$$\frac{2B}{3} = -2$$

$$B = \frac{-2 \times 3}{2}$$

$$B = -3$$

Similarly when $\Psi_3(\mathrm{x}) = 1 + Ax + Bx^2$ is orthogonal to $\Psi_2(\mathrm{x}) = x$, the integral of the product of these fuctions within (-1, 1) will be 0. Therefore,

$$\int_{-1}^{1} \left(\Psi_2(x) \times \Psi_3(x)\right) dx = 0$$

$$\int_{-1}^{1} \left(x \times (1 + Ax + Bx^2)\right) dx = 0$$

$$\int_{-1}^{1} (\mathrm{x} + Ax^2 + Bx^3) dx = 0$$

$$\left[\frac{x^2}{2} + A\frac{x^3}{3} + B\frac{x^4}{4}\right]_{-1}^{1} = 0$$

$$\frac{1}{2}\left(1^2 - (-1)^2\right) + \frac{A}{3}\left(1^3 - (-1)^3\right) + \frac{B}{3}\left(1^4 - (-1)^4\right) = 0$$

$$\frac{1}{2}(1 - 1) + \frac{A}{3}(1 + 1) + \frac{B}{4}(1 - 1) = 0$$

$$\frac{1}{2} \times 0 + \frac{A}{3} \times 2 + \frac{B}{3} \times 0 = 0$$

$$0 + \frac{2A}{3} + 0 = 0$$

$$A = 0$$

Therefore, $A = 0$ and $B = -3$